An FPTAS for KNAPSACK.

Theorem: The profit achieved by the algorithm is at least (1- ε)OPT.

Proof: Let S denote the set of items packed by the PTAS, and let S* denote the set of items packed in some optimal solution. Thus, $ALG=\Sigma_{i\in S} b_i$, $OPT = =\Sigma_{i\in S^*} b_i$.

The rounding could have increased each b_i by less than k, since ceiling increases the value of the fraction by less than 1.

$ALG = \sum_{i\in S} bi \geq \sum_{i\in S}(b'i - k) = \sum_{i\in S} b'i - |S|k = \sum_{i\in S} b' i - |S| \cdot \varepsilon B/n \geq \sum_{i\in S} b'i - \varepsilon B$

Due to the ceiling and since S is optimal for the rounded instance, it must be that: $\sum_{i\in S} b' i \geq \sum_{i\in S^*} bi$, and we can conclude: $ALG \geq \sum_{i\in S} b'i - \varepsilon B \geq \sum_{i\in S^*} bi - \varepsilon B$.

Finally, since $B \leq \sum_{i\in S^*} bi$ (packing only the element whose value is B is a valid solution), we have: $ALG \geq \sum_{i\in S^*} bi - \varepsilon B \geq (1-\varepsilon) \sum_{i\in S^*} bi = (1-\varepsilon)OPT$.

Any-fit Decreasing for Unit-Fraction Bin Packing

After being sorted in a non-increasing order, the input sequence has the form

$$W = \langle (\frac{1}{2})^{n2}, (\frac{1}{3})^{n3} \dots (\frac{1}{z})^{nz} \rangle$$

for some integers $z \geq 2$ and $ni \geq 0$ for $2 \leq i \leq z$.

Assume that AFD uses h full bins (filled to capacity 1) and h' non-full bins. Thus, N_AFD(W) = h'+h.

Claim: After packing all the items of size at least 1/k, there are at most k-1 non-full bins.

Proof: The proof is by induction on k. The base case is k=2. Clearly, the items of size 1/2 are packed in $\lceil n/2 \rceil$ bins, where only the last one may be non-full.

Assume that the claim holds before packing the items of size 1/k. That is, after packing the items of width at least 1/(k-1), there are at most k-2 non-full bins. Since items of size 1/k are first added to currently non-full bins that can accommodate them, it follows that only one bin that contains only items of size 1/k may not be full after all the 1/k-items are packed.

Suppose the last bin that AFD opened was opened for an item of width 1/z' where $z' \leq z$. By the above claim, at this stage, there are less than z' non-full bins. Since this is the last opened bin, it follows that $h' < z'$.

Furthermore, each of the first h'-1 non-full bins must contain items whose total width is greater than 1-(1/z'), because otherwise AFD would not open a new bin for 1/z'. By definition, the last non-full bin contains at least one item of size 1/z'. It follows that

$H(W) \geq h + (h'-1)(1-1/z')+1/z' = h+h'-1-(h'-2)/z' > h+h'-2$.

Since $\lceil H(W) \rceil$ is an integer, it must be at least h'+h-1. Thus, $N\_AFD(W) = h' + h \leq \lceil H(W) \rceil +1 \leq OPT+1$.