Applied Algorithms, Summer 2017
Homework 2,
Due date: XX/7/17

1. (25 pts.) In class we saw an optimal algorithm for the Planar Single Facility Rectilinear Distance MiniSum problem (slides 7-9). For each of the following algorithms, either prove that it is another optimal algorithm for the problem or give an example to show it is sub-optimal.

Algorithm A: Let x*= weighted average of x-coordinate of the requests ($= \Sigma w_i a_i / \Sigma w_i$).

Let y*= weighted average of y-coordinate of the requests ($= \Sigma w_i b_i / \Sigma w_i$).

Returen (x*,y*)

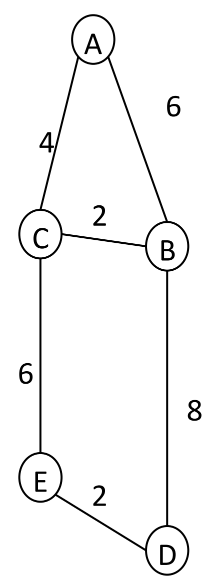
Algorithm B: Let x*= weighted median of x-coordinate of the requests (median of a sorted vector of length Σw_i in which a_i appears w_i times)

Let y*= weighted median of y-coordinate of the requests (median of a sorted vector of length Σw_i in which b_i appears w_i times)

If Σw_i is even, then select as median either the $\lfloor \Sigma w_i/2 \rfloor$ or the $\lceil \Sigma w_i/2 \rceil$ value.

Returen (x*,y*)

2. (25 pts.)
a. Solve the 1-center problem for the graph in the figure. Find both a single node center and an absolute center.
b. Solve the 1-median problem for the graph. Assume h(A)=3, h(B)=1, h(C)=2, h(D)= 2, h(E)=1.
Remark: when finding the distance matrix, you don't need to actually run a shortest path algorithm, it is OK to conclude the values from the figure.



3. (25 pts.) Given is a star graph (a tree with depth 1) consisting of a middle node v and n>1 surrounding vertices. It is known that all vertices have positive weights and all edges have positive lengths. Prove or give a counter example for each of the following claims:

a. If all edges have the same length (and vertices have arbitrary weights) then v is the 1-median.

b. If all vertices have the same weights (and edges have arbitrary lengths) then v is the 1-median.

c. A new vertex is added to the graph and is connected by a new edge to one of the <u>outer</u> vertices of the star. If all vertices (including the new one) have weight 1, and all the star edges (but not necessarily the new edge) have the same length, then any 1-median of the graph is also a 1-center.

4. (25 pts.) As a result of the next high-tech crisis, your company is closed and you choose a new career in a farm. You have n potatoes with weights $b_1, b_2, \ldots, b_n$ grams that you wish to pack into packages. Each package must contain at least L grams of potatoes so you won't get sued for false advertising. Your goal is to maximize the number of packages that you fill to L or more grams. Consider the obvious greedy algorithm that considers the potatoes in an arbitrary order, and adds potatoes to the same package until that package contain at least L grams of potatoes. Assume that no potato weighs more than L.

a. Show that this is a 2-approximation algorithm. That is, the algorithm fills at least 1/2 as many packages as optimal.

b. Given m, describe an instance with $n=\Theta(m)$ potatoes for which the 2-ratio is as tight as possible.

Solutions

1. In both parts (algorithms A and B) it is ok to analyze the x-coodinate only. As we saw in class, the objective function is can be separated, and an optimal solution is optimal for each coordinate separately.

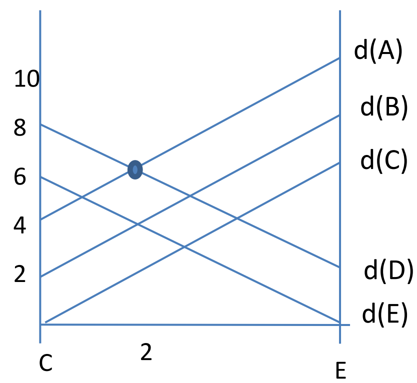
Algorithm A is not optimal. Exampe: Let x1=0 w1=10, x2=10, w2=1. The optimal solution is x*=0, its objective value is 10*0+1*10=10. The weighted average is at some positive point x'>0. Its objective value is 10x'+(10-x')=10+9x' >10.

Algorithm B is optimal. Assume that an optimal solution x' is not located in the median point as suggested by the algorithm. W.L.O.G, the weight of the points left of x' is higher by Δ than the weight right of it. By moving the solution to x'-ε, the objective function is reduced by Δε, contradicting the optimality of x'. If Δ=0 then it is possible to move x' left to the nearest point – this corresponds to the case that Σw_i is even, and x' was selected arbitrarily.

2.a Vertex 1-center: both B and C have the same minimal objective value (=8).

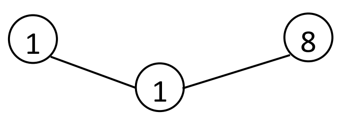
Absolute 1-center : Ploting the distance graph on the edge (CE), we get that the local center on this edge is 2 units from C, and its value is 6.

Given that the current candidate has value t=6, we use the theorem according to which it worth checking an edge (p,q) only if $t>(m(p)+m(q)-c(p,q))/2$. We get that no other edge should be considered.



2.b The median is C with objective value = 36.

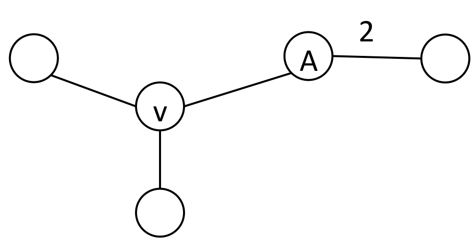
3.a. False. Consider for example n=2 with need weights as specified in the figure



b True: Assume all vertices have weight w. The objective value of the middle node is $J(v)= w\Sigma_{i=1}^{n}d(v,v_i)$. The objective value of any other node v_j is $J(v_k)= w(\Sigma_{i=1}^{n}d(v,v_i)+ (n-1)d(v,v_j))$. [note that v also needs to reach v_j]

Thus, $J(v)= J(v_k)-w(n-1)d(v,v_j))$, and v is a 1-median [if d(v,vj)=0 then both might be 1-medians].

c False. In the following graph v is the only 1-median (J(v)=6) but A is the only 1-center.



4.1 Let S be the total weight of the potatoes. Clearly, an optimal algorithm packs at most $\lfloor S/L \rfloor$ packages. Another simple observation is that the greedy algorithm never fills a package by more than 2L, therefore the greedy algorithm fills at least $\lfloor S/2L \rfloor$ packages.

The above is not sufficient to conclude a 2-ratio (try for example S=7, L=2 : $\lfloor 7/4 \rfloor / \lfloor 7/2 \rfloor = 1/3 < 1/2$. The calculation is a bit more complex: Assume S=xL+r for some r<L. If x is even then OPT fills at most x packages and Greedy fills at least x/2 packages, which implies a 2-ratio. If x is odd, say x=2y+1, then OPT fills at most 2y+1 packages and Greedy fills at least y+1 packages – because each of the first y packages holds at most 2L and there are enough potatoes for one additional bin (maybe less than 2L but more than L). In this case the ratio is (y+1)/2y > ½.

3.2 Given m, let ε be less than L/m. Consider an instance with m potatoes of weight L-ε followed by m potatoes of weigh ε. The greedy algorithm packs in each of the first m/2 packages two of the L-ε potatoes. The remaining ε potatoes are too small for filling a package.

An optimal solution packs m packages, each with one large and one small potato. The ratio is m/(m/2) =2.