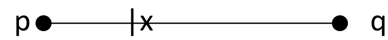
Facility Location.

Theorem (p.33) : For the local center, x_e , on an edge (p,q),

$$m(x_e) \geq \frac{m(p)+m(q)-c(p,q)}{2},$$ where, c(p, q) denotes the length of (p,q).

Proof: Consider a point x on e with distance x from p.



For x=0, the point is p. For x=c(p,q) the point is q.

d(x,p)=x and d(x,p)=c(p,q)-x.

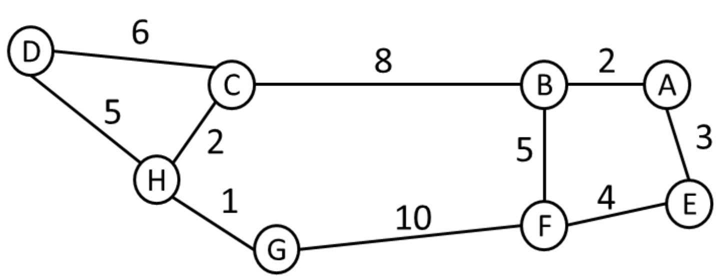
Therefore, we have m(p) ≤ m(x)+x and m(q) ≤ m(x)+c(p,q)-x, since it is always possible to reach p by a path to x plus d(p,x), and it is always possible to reach q by a path to x plus d(x,q).

Summing up the above inequations, we get m(q)+m(p) ≤ 2m(x)+x –c(p,q)-x =2m(x)+c(p,q).

The above is valid for every x, in particular, for $x=x_e$.

By switching sides, we get $m(x_e) \geq \frac{m(p)+m(q)-c(p,q)}{2}$

Example of 2-approximation to k-center (p.37) Let k=3



Assume that A is selected as a first node.

The furthest node is D – since d(D,A)=16. So D is added.

The next furthest node is F – since d(F,{A,B})=7

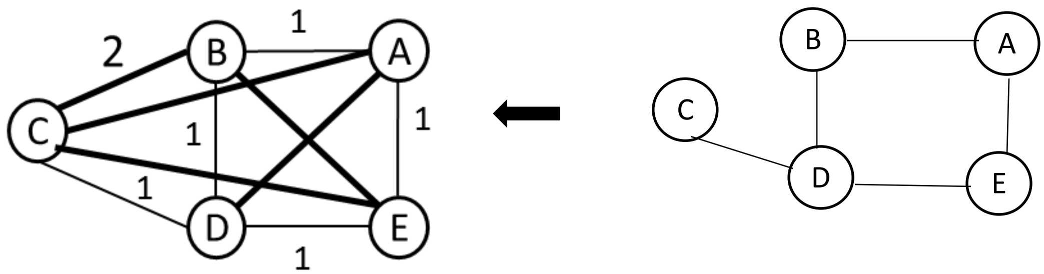
X_3 = {A,D,F}. The value of this solution is 6 – d(C,X_3)=d(G,X_3)=6.

A better solution is {B,F,H} – its value is 5.

For k=2, the algorithm halts with {A,D}, d(f,X2})=7.

OPT(k=2)={B,H}, value =5 (achieved by E and F).

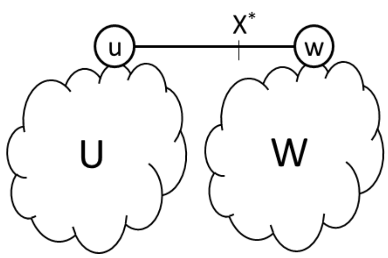
Example - no (2-ε)-approximation (p. 40).



Hakimi's Theorem: At least one optimal set of k-medians exist solely on the nodes of G (p. 43).

Proof: Assume that k=1.

Let x* be the optimal 1-median. If x* is a node – we are done. Otherwise, x*is located on some edge (u,w). Split the graph's nodes into two disjoint sets V=U∪W such that v∈U if and only if a shortest path from v to x* passes through v (otherwise, v∈W)



Compare $\sum_{v\in U} h(v)$ with $\sum_{v\in W} h(v)$. Assume w.l.o.g. that $\sum_{v\in U} h(v) \geq \sum_{v\in W} h(v)$.
We show that x* can be replaced by the node u without hurting the objective function value.

$$J(x^*) = \sum_{v\in V} h(v)d(v,x^*) = \sum_{v\in U} h(v)[d(v,u)+d(u,x^*)] + \sum_{v\in W} h(v)d(v,x^*) =$$

$$\sum_{v\in U} h(v)d(v,u) + \sum_{v\in U} h(v)d(u,x^*) + \sum_{v\in W} h(v)d(v,x^*) \geq$$

assumption $$\geq \sum_{v\in U} h(v)d(v,u) + \sum_{v\in W} h(v)d(u,x^*) + \sum_{v\in W} h(v)d(v,x^*) =$$

$$\geq \sum_{v\in U} h(v)d(v,u) + \sum_{v\in W} h(v)[d(u,x^*)+d(v,x^*)] \geq$$ Triangle inequality

$$\geq \sum_{v\in U} h(v)d(v,u) + \sum_{v\in W} h(v)d(u,v) = \sum_{v\in V} h(v)d(u,v) = J(u).$$

For k>1, a similar approach works for every facility located along an edge.