Theorem: Every congestion game is a potential game.

Proof: For a given profile *a*, define the potential of *a* to be $\Phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} c_j(k)$. Assume that *a* is not a NE and that player i performs an improving step. All other players do not change their strategies. Denote by b the new profile, and by Δu_i the change in i's cost.

$\Delta u_i = \sum_{j \in b_i - a_i} c_j(n_{j(a)} + 1) - \sum_{j \in a_i - b_i} c_j(n_{j(a)})$.

Note that the change in the potential is exactly equal to Δu_i.

Thus, given that i's cost decreases, we conclude that $\Phi(b) < \Phi(a)$. That is, the potential function strictly decreases along the BRD process.

Note: When $\Delta u_i = \Delta\Phi$ we say that Φ is an exact potential function.

NE calculation in Symmetric Network Formation Congestion Games.

Let f be a flow of value n. Since all capacities are 1, it is possible to decompose the flow into n edge-disjoint s-t paths. Each such path is a possible strategy of some player. Denote by a(f) the profile induced by the flow.

Claim: Let f^* be a min-cost max-flow in the network. The capacity of (s',s) implies that f^* has value n. Consider $a(f^*)$. Since f^* achieves min-cost, for every pair of nodes u and v, if an edge (u,v) is used by $n_e(a)$ players, then, since f^* achieves min-cost, the $n_e(a)$ edges that are used in the flow are the cheapest ones, and the total flow cost is exactly $\Phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a(f*))} c_j(k)$.

We conclude that a min-cost flow corresponds to a profile with minimum potential – which must be a NE.