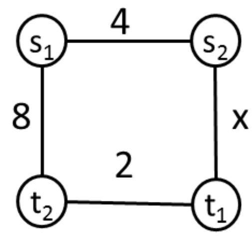
A Network Formation Game – Example.



The figure presents a 2-players network formation game.
The edges are undirected and fair cost-sharing is applied.
The objective of player i=1,2 is an <si-ti>-path.

For every value of $x \geq 0$ determine

1. The social optimum.
2. The routings that form a NE
3. The PoA and the PoS.

Solution: Every player has two strategies, therefore there are four possible profiles (routings):

Profile	strategies	Cost 1	Cost 2	Total
R1	↘ ↘	4+x/2	x/2+2	x+6
R2	↘ ↙	2+x	10	x+12
R3	↙ ↘	9	x+1	x+10
R4	↙ ↙	6	8	14

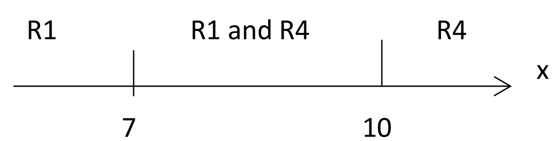
1. Calculating the social optimum: The minimum total cost is either x+6 or 14.

For $x \leq 8$, SO=x+6. For x>8, SO=14.

2. Calculating stable profiles:

For each routing, we write the stability conditions for each of the players. A routinh is stable if both conditions hold simultaneously.

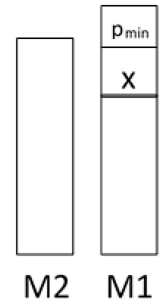
R1 is a NE if $4+x/2 \leq 9$ and $x/2+2 \leq 10$
$x \leq 10$ and $x \leq 16$.
R2 is a NE if $2+x \leq 6$ and $10 \leq x/2+2$
$x \leq 4$ and $x \geq 16$. **NEVER**
R3 is a NE if $9 \leq 4+x/2$ and $x+1 \leq 8$
$10 \leq x$ and $x \leq 7$. **NEVER**
R4 is a NE if $6 \leq 2+x$ and $8 \leq x+1$
$4 \leq x$ and **$7 \leq x$.**



3. Equilibrium inefficiency:

For x<7, R1 is the only one NE.
PoA=PoS= (x+6)/(x+6)=1
For x>10, R4 is the only NE.
PoA=PoS=14/14=1

For $7 \leq x \leq 8$ PoS=min{(x+6),14}/(x+6)=(x+6)/(x+6)=1. PoA= max{(x+6),14}/(x+6)=14/(x+6).
For $8 \leq x \leq 10$ PoS=min{(x+6),14}/14=(x+6)/(x+6)=1. PoA= max{(x+6),14)/14=x+6/14.



Theorem: LPT algorithm for job scheduling produces a NE schedule.

Proof:

Claim: If a job of length x is assigned on a machine M1 with load L1 (including x), then for every other machine, M2, with load L2, it holds that L1 $\leq$ L2+x.

Remark: L1 and L2 are the final loads of M1 and M2.

Proof: Case 1: L1$\leq$ L2, then clearly L1 $\leq$ L2+x.

Case 2: L2<L1. Let $p_{min} \leq x$ be the length of the last job assigned on M1. Denote by L1t, L2t the loads on M1 and M2 before the assignment of p_{min}. By LPT rule, given that p_{min} was assigned on M1, we have L1t $\leq$ L2t. After the assignment of pmin, the load on M2 could only increase. Therefore L2 $\geq$ L2t. Also, by definition of p_{min}, L1=L1t+p_{min}. Combining the above inequalities, we get L1 ≤ L2t+p_{min} ≤ L2t+x ≤ L2+x.

The claim implies that a migration of a job of length x from M1 to M2 is not beneficial. Since the choice of x,M1 and M2 is arbitrary. Any schedule produced by LPT is a NE.

Theorem: Every job scheduling game has a strong NE.

Proof: We characterize a schedule by a vector of length m (L1,…,Lm) such that L1≥L2…≥Lm. That is, Li is the load on the i-th loaded machine.

Definition: Given two vectors X=(x1,…,xm) and Y=(y1,…,ym) we say that x is lexicographically smaller than y, (denoted $x \prec y$) if there exists an i such that xi<yi and for all j<I, we have xj=yj. For example (8,5,3) $\prec$ (8,6,2).

Claim: Let p be a profile corresponding to a schedule achieving the minimal lexicographic load vector, then p is a SE (in particular, it achieves minimum makespan).

Remark: It is easy to see that this schedule is a NE – since every beneficial move corresponds to reducing the load vectors.

Proof: Assume by contradiction that some coalition exists. Let Γ be a coalition consisting of a minimal number of players. Let M(Γ) be the set of machines on which the jobs of Γ are assigned. We show that in every coordinated deviation of Γ, at least one job leaves and at least one job joins every machine in M(Γ). Consider a machine M $\in$ M(Γ).

-if no job leaves M, then p is not a NE – since every job migrating into M benefits.

-if no job joins M, then let j be a job that leaves M. Such a job exists since M $\in$ M(Γ). Note that Γ-{j} is also a coalition. Contradicting the minimality of Γ.

Let Mi be the most loaded machine in M(Γ). Its load in p is Li. By the above claim, some job joins Mi from some Mk. Therefore, the load on Mi after the deviation is less than Lk. Also Lk $\leq$ Li (Mi is most loaded). We conclude that the load on the most loaded machine reduces. Also, since the deviation is beneficial to all the coalition members, no other machine has load Li.

We conclude that the resulting load vector is lexicographically smaller – contradicting the definition of p as having the minimal lexicographic load vector.